# Re-expressing multi-particle states using creation/annihilation operators.

Consider an operator with the following operation:

$$a^+(\varphi)\,|u_1, u_2, \ldots u_n\rangle$$
$$= |\varphi, u_1, u_2, \ldots u_n\rangle$$

i.e. $a^+(\varphi)$ 'creates' a particle in a state $|\varphi\rangle$. Note that the new-state has now $\underline{n+1}$ particles.

Consider two such operators $a^+(\varphi_1)$, $a^+(\varphi_2)$

$$a^+(\varphi_1)\,a^+(\varphi_2)\,|u_1, u_2, \ldots u_n\rangle$$
$$= |\varphi_1, \varphi_2, u_1, u_2 \ldots u_n\rangle$$

On the other hand,

$$a^+(\varphi_2)\,a^+(\varphi_1) = |\varphi_2, \varphi_1, u_1, \ldots u_n\rangle$$

If the state corresponds to bosons / fermions

$$| \varphi_1, \varphi_2, u_1, \dots u_n \rangle$$

$$= \pm | \varphi_2, \varphi_1, u_1, \dots u_n \rangle$$

$$\Rightarrow \quad a^\dagger(\varphi_1)\, a^\dagger(\varphi_2) = \xi\, a^\dagger(\varphi_2)\, a^\dagger(\varphi_1)$$

where $\xi = -1$ for fermions
$\phantom{where \xi} = +1$ for bosons.

Taking the hermitian conjugate,

$$a(\varphi_1)\, a(\varphi_2) = \xi\, a(\varphi_2)\, a(\varphi_1)$$

Is there any relation between $a^\dagger(\varphi_1) a(\varphi_2)$ and $a(\varphi_2)\, a^\dagger(\varphi_1)$ ?

To see this, we first need to calculate the action of $a(\varphi)$ on a state.

$$\langle v_1, v_2, \dots v_{n-1} | a(\varphi) | u_1, \dots u_n \rangle$$

$$= \langle u_1, \dots u_n | a^{+}(\varphi) | v_1, \dots v_{n-1} \rangle^{*}$$

$$= \langle u_1, \dots u_n | \varphi, v_1, \dots v_{n-1} \rangle^{*}$$

$$= \sum_{P_n} \xi^{P_n} \langle \varphi | u_{p(1)} \rangle \langle v_1 | u_{p(2)} \rangle \dots \langle v_{n-1} | u_{p(n)} \rangle$$

($P_n$ denotes permutation of n objects)

$$= \sum_{p(1)} \langle \varphi | u_{p(1)} \rangle \sum_{p(2) \dots p(n)} \xi^{P_n} \langle v_1 | u_{p(2)} \rangle \dots \langle v_{n-1} | u_{p(n)} \rangle$$

$$= \sum_{k} \langle \varphi | u_k \rangle \; \xi^{k-1} \langle v_1 \dots v_{n-1} | u_1 \dots (\text{no } u_k) \dots u_n \rangle$$

$$\Rightarrow \quad a(\varphi) | u_1, \dots u_n \rangle$$

$$= \sum_{k} \xi^{k-1} \langle \varphi | u_k \rangle \, | u_1 \dots (\text{no } u_k) \dots u_n \rangle$$

Using this,

$$a(\varphi_1)\, a^+(\varphi_2)\, |u_1 \cdots u_n\rangle$$

$$= a(\varphi_1)\, |\varphi_2, u_1 \cdots u_n\rangle$$

$$= \langle \varphi_1 | \varphi_2 \rangle\, |u_1 \cdots u_n\rangle$$

$$+ \sum_{k=1}^{n} \langle \varphi_1 | u_k \rangle\, \xi^k\, |\varphi_2, u_1 \cdots (\text{no } u_k) \cdots u_n\rangle$$

Similarly,

$$a^+(\varphi_2)\, a(\varphi_1)\, |u_1, \cdots u_n\rangle$$

$$= a^+(\varphi_2) \sum_k \xi^{k-1} \langle \varphi_1 | u_k \rangle\, |u_1 \cdots (\text{no } u_k) \cdots u_n\rangle$$

$$= \sum_k \xi^{k-1} \langle \varphi_1 | u_k \rangle\, |\varphi_2\, u_1 \cdots (\text{no } u_k) \cdots u_n\rangle$$

$$\Rightarrow \boxed{a(\varphi_1)\, a^+(\varphi_2) - \xi\, a^+(\varphi_2)\, a(\varphi_1) = \langle \varphi_1 | \varphi_2 \rangle}$$

**Examples:**

If $\vec{x}$ denotes real-space coordinate and $\vec{p}$ the momentum.

$$a(\vec{x})\, a^{+}(\vec{x}') - s\, a^{+}(\vec{x}')\, a(\vec{x})$$

$$= \langle \vec{x} | \vec{x}' \rangle = \delta^{d}(\vec{x} - \vec{x}')$$

where $d$ is the spatial dimension.

$S = +1$ for bosons and $-1$ for fermions (we will use the symbol $s$ instead of $\zeta$ from now on).

$$a(\vec{p})\, a^{+}(\vec{p}') - s\, a^{+}(\vec{p}')\, a(\vec{p})$$

$$= \langle \vec{p} | \vec{p}' \rangle = (2\pi)^{d}\, \delta^{d}(\vec{p} - \vec{p}')$$

why the factor of $(2\pi)^d$? This is due to the fact that

$$a^{+}(x) = \int \frac{d^d p}{(2\pi)^d}\, a^{+}(\vec{p})\, e^{-i\vec{p}\cdot\vec{x}}.$$